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Rate of Convergence in a Class of Singular Perturbations*

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Let V_ϵ, W_ϵ, W and X be Hilbert spaces $(0 < \epsilon \leqslant 1)$ with $V_\epsilon \subset W_\epsilon \subset W \subset X$ algebraically and topologically, each space being dense in the one that follows it. For each $t \in [0, T]$ let $a_\epsilon(t; u, v)$, $b_\epsilon(t; u, v)$ and $b(t; u, v)$ be continuous sesquilinear forms on V_ϵ, W_ϵ and W, respectively, which satisfy certain ellipticity conditions. Consider the two equations $a_\epsilon(t; u_\epsilon', v) + b_\epsilon(t; u_\epsilon, v) = \langle f_\epsilon, v\rangle$ $(v \in V_\epsilon)$ and $(u', v)_X + b(t; u, v) = \langle f, v\rangle$ $(v \in W)$. Estimates are obtained on the rate of convergence of u_ϵ to u, assuming $a_\epsilon(t; u, v) \to (u, v)_X$ and $b_\epsilon(t; u, v) \to b(t; u, v)$ in an appropriate sense. These results are then applied to singular perturbation of a class of parabolic boundary value problems.

1. INTRODUCTION

Let V_ϵ, W_ϵ, W and X be Hilbert spaces (ϵ being a real parameter, $\epsilon \in (0, 1]$ for example) with

$$V_\epsilon \subset W_\epsilon \subset W \subset X. \tag{1.1}$$

The inclusions are both algebraic and topological and each space is assumed dense in the one that follows it. The norm and scalar product in X are denoted by $|\cdot|$ and $(\cdot, \cdot)$, respectively; norms and scalar products in the other spaces are indicated with subscripts, for example, $|\cdot|_{V_\epsilon}$ and $(\cdot, \cdot)_{V_\epsilon}$.

We identify X with its antidual. Then we may write

$$X \subset W' \subset W_\epsilon' \subset V_\epsilon',$$

where the prime means antidual. If x belongs to one of the spaces in (1.1) and f to its antidual, we denote their scalar product in the antiduality by $\langle f, x\rangle$; this coincides with (f, x) when $f \in X$.

t will denote a real variable taken from an interval $[0, T]$. For convenience we assume $T < \infty$; only minor modifications are required

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if $T = +\infty$. For each ϵ and t let $a_\epsilon(t; u, v)$, $b_\epsilon(t; u, v)$ and $b(t; u, v)$ be continuous sesquilinear forms on V_ϵ, W_ϵ and W, respectively. The antilinear forms $v \to a_\epsilon(t; u, v)$, $v \to b_\epsilon(t; u, v)$ and $v \to b(t; u, v)$ define bounded linear operators $\mathscr{A}_\epsilon(t) \in \mathscr{L}(V_\epsilon, V_\epsilon')$, $\mathscr{B}_\epsilon(t) \in \mathscr{L}(W_\epsilon, W_\epsilon')$ and $\mathscr{B}(t) \in \mathscr{L}(W, W')$, respectively, such that

$$a_\epsilon(t; u, v\rangle = \langle \mathscr{A}_\epsilon(t)u, v\rangle, \qquad u, v \in V_\epsilon,$$

$$b_\epsilon(t; u, v) = \langle \mathscr{B}_\epsilon(t)u, v\rangle, \qquad u, v \in W_\epsilon,$$

$$b(t; u, v) = \langle \mathscr{B}(t)u, v\rangle, \qquad u, v \in W.$$

We consider the two problems

$$\mathscr{A}_\epsilon(t)\, u_\epsilon' + \mathscr{B}_\epsilon(t)\, u_\epsilon = f_\epsilon(t), \qquad 0 < t < T, \qquad u_\epsilon(0) = x_\epsilon, \tag{1.2}$$

$$u' + \mathscr{B}(t)u = f(t), \qquad 0 < t < T, \qquad u(0) = x, \tag{1.3}$$

where $' = d/dt$ in (1.2) and (1.3). The purpose of this paper is to estimate the rate of convergence of u_ϵ to u, assuming $\mathscr{A}_\epsilon(t) \to I$ (identity on W'), $\mathscr{B}_\epsilon(t) \to \mathscr{B}(t)$, $f_\epsilon \to f$ and $x_\epsilon \to x$ in some appropriate fashion.

Under the hypotheses we shall impose, (1.2) serves as a model for various physical problems (for example, problems involving non-Newtonian fluids, soil mechanics, heat conduction; see, e.g. [8, 11] for relevant literature). The most common occurrence in these applications is $V_\epsilon = W_\epsilon = W$, $a_\epsilon(t; u, v) = (u, v) + \epsilon a(u, v)$, $a(u, v) = \overline{a(v, u)}$, $b_\epsilon(t; u, v) \equiv b(u, v)$ with $a(u, v)$ and $b(u, v)$ both W-elliptic. Convergence and rate of convergence for this special case have been considered in [8] and, for the even more specialized situation $W = H_0^1(\Omega)$ and $b(u, v) = \overline{b(v, u)}$, in [11] by methods entirely different from those employed here. Even as applied to these special cases, the present results are much sharper in most respects although, unlike those obtained here, the results of [8] apply to perturbations of equations of Schroedinger type.

In order to derive the necessary estimates, we shall make use of some aspects of the theory of intermediate spaces. There are a number of papers which also employ interpolation theory to derive estimates somewhat related to those given here. In this connection we cite in particular the papers [2, 3, 4, 7].

The next section is devoted to preliminary material while our principal results for the abstract problems (1.2) and (1.3) are given in Section 3. In Section 4 these results are applied to singular perturbation of a class of parabolic boundary value problems.

2. Preliminaries

The forms $a_\epsilon(t; u, v)$, $b_\epsilon(t; u, v)$ and $b(t; u, v)$ are assumed to satisfy[1] the following.

(2.1) For $u, v \in V_\epsilon$ we have

$$a_\epsilon(t; u, v) = \overline{a_\epsilon(t; v, u)};$$

the map $t \to a_\epsilon(t; u, v)$ is differentiable, the derivative $a_\epsilon'(t; u, v)$ is measurable on $[0, T]$ and

$$| a_\epsilon(t; u, v)| + | a_\epsilon'(t; u, v)| \leqslant M_\epsilon | u |_{V_\epsilon} | v |_{V_\epsilon} + M | u | | v |,$$

$$a_\epsilon(t; v, v) \geqslant \alpha_\epsilon | v |_{V_\epsilon}^2 + \alpha | v |^2,$$

where α_ϵ, α, M_ϵ and M are positive constants.

(2.2) For $u, v \in W_\epsilon$ the map $t \to b_\epsilon(t; u, v)$ is measurable on $[0, T]$ and

$$| b_\epsilon(t; u, v)| \leqslant N_\epsilon | u |_{W_\epsilon} | v |_{W_\epsilon} + N | u |_W | v |_W ,$$

$$\operatorname{Re} b_\epsilon(t; v, v) \geqslant \beta_\epsilon | v |_{W_\epsilon}^2 + \beta | v |_W^2 ,$$

where β_ϵ, β, N_ϵ and N are positive constants.

(2.3) For $u, v \in W$ the map $t \to b(t; u, v)$ is measurable on $[0, T]$ and

$$| b(t; u, v)| \leqslant K | u |_W | v |_W ,$$

$$\operatorname{Re} b(t; v, v) \geqslant \gamma | v |_W^2$$

with positive γ and K.

The form $a_\epsilon(t; u, v)$ is compared with (u, v) and $b_\epsilon(t; u, v)$ compared to $b(t; u, v)$ in the following way.

(2.4) Set

$$r_\epsilon(t; u, v) = a_\epsilon(t; u, v) - (u, v), \qquad u, v \in V_\epsilon ,$$

$$s_\epsilon(t; u, v) = b_\epsilon(t; u, v) - b(t; u, v), \qquad u, v \in W_\epsilon .$$

Then

$$| r_\epsilon(t; u, v)| + | r_\epsilon'(t; u, v)| \leqslant \rho_\epsilon | u |_{V_\epsilon} | v |_{V_\epsilon} ,$$

$$| s_\epsilon(t; u, v)| \leqslant \sigma_\epsilon | u |_{W_\epsilon} | v |_{W_\epsilon} ,$$

[1] In order to simplify certain measurability considerations, we suppose V_ϵ, W_ϵ, W and X are separable, although this is not essential.

where

$$\frac{\rho_\epsilon}{\alpha_\epsilon} = 0(1), \qquad \frac{\sigma_\epsilon}{\beta_\epsilon} = 0(1) \qquad \text{as} \qquad \epsilon \to 0_+ .$$

It follows from (2.1) and (2.4) that we may assume

$$\frac{M_\epsilon}{\alpha_\epsilon} = 0(1) \qquad \text{as} \quad \epsilon \to 0_+ .$$

A few ideas concerning particular intermediate spaces of J. Peetre will be needed in the next section (see [1, Chapter 3] for details). Let Y and Z be *Banach* spaces, both of which are contained algebraically and topologically in some locally convex topological vector space, and let $Y + Z$ denote their algebraic sum. $Y + Z$ is a Banach space under the norm

$$| x |_{Y+Z} = \inf_{x=y+z} (| y |_Y + | z |_Z), \qquad y \in Y, \quad z \in Z.$$

For each $x \in Y + Z$ and $t > 0$ set

$$K(t; x) = \inf_{x=y+z} (| y |_Y + t \mid z |_Z), \qquad y \in Y, \quad z \in Z.$$

If $0 < \theta < 1$, the intermediate space $[Y, Z]_\theta$ is defined as follows:

$$[Y, Z]_\theta = \{x : x \in Y + Z, t^{-\theta-1/2} K(t, x) \in L^2(0, \infty)\}.$$

It is a Banach space under the norm

$$| x |_{[Y,Z]_\theta} = \left[\int_0^\infty (t^{-\theta-1/2} K(t, x))^2 \, dt\right]^{1/2}.$$

The spaces $[Y, Z]_\theta$ have the *interpolation property*: let $\tilde{Y}$, $\tilde{Z}$ be a second pair of Banach spaces with properties analogous to those of Y, Z, and let $R \in \mathscr{L}(Y + Z, \tilde{Y} + \tilde{Z})$ such that $R_Y \in \mathscr{L}(Y, \tilde{Y})$ and $R_Z \in \mathscr{L}(Z, \tilde{Z})$ where the subscripts indicate restrictions. Let

$$M_0 = \| R_Y \|_{L(Y,\tilde{Y})} , \qquad M_1 = \| R_Z \|_{L(Z,\tilde{Z})} .$$

Then the following is true.

PROPOSITION 2.1. *The restriction of R to $[Y, Z]_\theta$ is a bounded linear mapping of $[Y, Z]_\theta$ into $[\tilde{Y}, \tilde{Z}]_\theta$ and the operator norm M_θ of this restriction satisfies*

$$M_\theta \leqslant M_0^{1-\theta} M_1^{\theta}.$$

Suppose now that Y and Z are Hilbert spaces with $Y \subset Z$ algebraically and topologically such that Y is dense in Z. Then $[Y, Z]_\theta$ coincides (with equivalent norm) with the intermediate space by quadratic interpolation between Y and Z of index θ introduced by Lions ([9]; c.f. [10, Chapter 1]). Set

$$H^1(0, T; Y) = \{u: u, u' \in L^2(0, T; Y)\}.$$

It is a Hilbert space under the norm

$$| u |_{H^1(0,T;Y)} = \left[\int_0^T (| u(t)|_Y^2 + | u'(t)|_Y^2)\, dt\right]^{1/2}.$$

$H^1(0, T; Z)$ is defined analogously.

PROPOSITION 2.2. *For* $0 < \theta < 1$,

$$[H^1(0, T; Y), H^1(0, T; Z)]_\theta = H^1(0, T; [Y, Z]_\theta).$$

The proof of this proposition can proceed exactly as does the proof of Theorem 1.14.2 of [10].

3. RATE OF CONVERGENCE

With hypothesis (2.3), it is well known [10, Chapter 3] that (1.3) has a unique solution $u \in L^2(0, T; W) \cap H^1(0, T; W')$ provided $f \in L^2(0, T; W')$ and $x \in H$. Furthermore, (2.1) and (2.2) are (more than) sufficient to assure the existence of a unique solution $u_\epsilon \in H^1(0, T; V_\epsilon)$ of (1.2), assuming $f_\epsilon \in L^2(0, T; V_\epsilon')$ and $x_\epsilon \in V_\epsilon$. [Indeed, set $\mathscr{V}_\epsilon = L^2(0, T; V_\epsilon)$, $\Lambda_\epsilon = -d/dt$ with $D(\Lambda_\epsilon) = \{u : u, u' \in \mathscr{V}_\epsilon, u(0) = 0\}$ and, for $u \in \mathscr{V}_\epsilon$, $(A_\epsilon u)(t) = -\mathscr{A}_\epsilon^{-1}(t)\, \mathscr{B}_\epsilon(t)\, u(t)$. Then $A_\epsilon \in \mathscr{L}(\mathscr{V}_\epsilon, \mathscr{V}_\epsilon)$ and Λ_ϵ is the generator of a (C_0)-semigroup of contractions on $\mathscr{V}_\epsilon$. By a well known result on perturbation of semigroups (c.f. [6, Chapter 12]) $\Lambda_\epsilon + A_\epsilon$ on $D(\Lambda_\epsilon)$ is the generator of a (C_0)-semigroup of bounded linear operators on $\mathscr{V}_\epsilon$. Thus, $(\Lambda_\epsilon + A_\epsilon - kI)^{-1}$ exists for all sufficiently large positive values of k as an everywhere defined, bounded linear operator on $\mathscr{V}_\epsilon$. This proves existence of a unique solution of (1.2) (and also its continuous dependence on f_ϵ) provided $\mathscr{B}_\epsilon$ is replaced by $\mathscr{B}_\epsilon + k\mathscr{A}_\epsilon$ and $x_\epsilon = 0$. But these are not essential restrictions, as is easily seen.]

Concerning the data in (1.2) and (1.3) we shall, henceforth, assume

$$f \text{ and } f_\epsilon \in L^2(0, T; W'), \qquad x_\epsilon \in V_\epsilon \qquad \text{and} \qquad x \in W. \tag{3.1}$$

The following describes the behavior of $u_\epsilon - u$ as $\epsilon \to 0_+$.

THEOREM 3.1. *Assume* (2.1)–(2.4), (3.1) *and that* $u(\cdot) - x \in [Y_\epsilon, Z]_\theta \equiv [H^1(0, T; V_\epsilon), L^2(0, T; W) \cap H^1(0, T; X)]_\theta$ *for some* $\theta \in (0, 1)$. *Then*

$$\begin{aligned} &| u_\epsilon - u |_{L^\infty(0,T;X)} + | u_\epsilon - u |_{L^2(0,T;W)} \\ &\quad \leqslant C\{[\max(\rho_\epsilon/\sqrt{\alpha_\epsilon}, \sigma_\epsilon/\sqrt{\beta_\epsilon})]^{1-\theta} \cdot | u - x |_{[Y_\epsilon, Z]_\theta} \\ &\quad\quad + | f_\epsilon - f |_{L^2(0,T;W')} + | x_\epsilon - x |_W + (\sigma_\epsilon/\sqrt{\beta_\epsilon}) | x_\epsilon |_{W_\epsilon}\} \end{aligned}$$

where C does not depend on ϵ, u or θ.

Remark. $L^2(0, T; W) \cap H^1(0, T; X)$ is the Hilbert space $\{u : u \in L^2(0, T; W), u' \in L^2(0, T; X)\}$ with the norm defined by

$$(| u |^2_{L^2(0,T;W)} + | u' |^2_{L^2(0,T;X)})^{1/2}.$$

The regularity hypothesis on u is difficult to verify in practice because of the problem of finding a "concrete" characterization of $[Y_\epsilon, Z]_\theta$. In this respect the following corollary is more satisfactory.

COROLLARY 3.1. *Assume* (2.1)–(2.4), (3.1) *and that* $u \in H^1(0, T; [V_\epsilon, W]_\theta)$ *for some* $\theta \in (0, 1)$. *Then*

$$\begin{aligned} &| u_\epsilon - u |_{L^\infty(0,T;X)} + | u_\epsilon - u |_{L^2(0,T;W)} \\ &\quad \leqslant C\{[\max(\rho_\epsilon/\sqrt{\alpha_\epsilon}, \sigma_\epsilon/\sqrt{\beta_\epsilon})]^{1-\theta} \cdot [| u |_{H^1(0,T;[V_\epsilon, W]_\theta)} + | x |_{[V_\epsilon, W]_\theta}] \\ &\quad\quad + | f_\epsilon - f |_{L^2(0,T;W')} + | x_\epsilon - x |_W + (\sigma_\epsilon/\sqrt{\beta_\epsilon}) | x_\epsilon |_{W_\epsilon}\} \end{aligned}$$

where C does not depend on ϵ, u or θ.

Remark. This corollary is especially useful when there is a Hilbert space $V_0 \subset V_\epsilon$ for all $\epsilon \in (0, 1]$ (algebraically, topologically with V_0 dense in V_ϵ). Then $[V_0, W]_\theta \subset [V_\epsilon, W]_\theta$ with continuous injection, and, therefore, one may replace $[V_\epsilon, W]_\theta$ everyplace that it appears in the corollary by $[V_0, W]_\theta$. We also note that because of (2.4), $\rho_\epsilon/\sqrt{\alpha_\epsilon}$ (respectively, $\sigma_\epsilon/\sqrt{\beta_\epsilon}$) converges to zero at least as fast as $\sqrt{\alpha_\epsilon}$ (respectively, $\sqrt{\beta_\epsilon}$).

Proof of Corollary 3.1. Since $H^1(0,T; W) \subset L^2(0,T; W) \cap H^1(0,T; X)$ with continuous injection we have, in view of Proposition 2.2,

$$H^1(0, T; [V_\epsilon, W]_\theta) \subset [H^1(0, T; V_\epsilon), L^2(0, T; W) \cap H^1(0, T; X)]_\theta$$

with continuous injection, from which the corollary follows.

Proof of Theorem 3.1. We first reduce the problem of estimating $u_\epsilon - u$ to one in which $f_\epsilon = f$ and $x_\epsilon = x = 0$. To do this set

$$\tilde{u}_\epsilon = u_\epsilon - x_\epsilon, \qquad v = u - x.$$

Then $\tilde{u}_\epsilon$ is the unique solution in $H^1(0, T; V_\epsilon)$ of

$$\mathscr{A}_\epsilon(t)\, \tilde{u}_\epsilon' + \mathscr{B}_\epsilon(t)\, \tilde{u}_\epsilon = f_\epsilon(t) - \mathscr{B}_\epsilon(t)\, x_\epsilon, \qquad \tilde{u}_\epsilon(0) = 0,$$

and v is the unique solution in $L^2(0, T; W) \cap H^1(0, T; W')$ of the problem

$$v' + \mathscr{B}(t)v = f(t) - \mathscr{B}(t)x, \qquad v(0) = 0. \tag{3.2}$$

Note that, by hypothesis, $v \in [Y_\epsilon, Z]_\theta$.

Let v_ϵ be the unique solution in $H^1(0, T; V_\epsilon)$ of

$$\mathscr{A}_\epsilon(t)\, v_\epsilon' + \mathscr{B}_\epsilon(t)\, v_\epsilon = f(t) - \mathscr{B}(t)x, \qquad v_\epsilon(0) = 0. \tag{3.3}$$

We proceed to estimate $\tilde{u}_\epsilon - v_\epsilon \equiv w_\epsilon$. For every $\xi \in V_\epsilon$ and almost all $t \in [0, T]$ we have

$$a_\epsilon(t; w_\epsilon', \xi) + b_\epsilon(t; w_\epsilon, \xi) = \langle f_\epsilon(t) - f(t), \xi\rangle - [b_\epsilon(t; x_\epsilon, \xi) - b(t; x, \xi)].$$

Let $\xi = w_\epsilon(t)$, take real parts and integrate from 0 to t to obtain, in view of (2.1)–(2.3)

$$\begin{aligned}
&(\alpha_\epsilon/2)\,|\, w_\epsilon(t)|_{V_\epsilon}^2 + (\alpha/2)\,|\, w_\epsilon(t)|^2 + \beta_\epsilon \,|\, w_\epsilon\,|_{\mathscr{W}_\epsilon(t)}^2 + \beta\,|\, w_\epsilon\,|_{\mathscr{W}(t)}^2 \\
&\quad \leqslant (M_\epsilon/2)\,|\, w_\epsilon\,|_{\mathscr{V}_\epsilon(t)}^2 + (M/2)\,|\, w_\epsilon\,|_{\mathscr{X}(t)}^2 + |\, f_\epsilon - f\,|_{\mathscr{W}'(t)}\,|\, w_\epsilon\,|_{\mathscr{W}(t)} \\
&\qquad + \sigma_\epsilon \sqrt{T}\,|\, x_\epsilon\,|_{W_\epsilon}\,|\, w_\epsilon\,|_{\mathscr{W}_\epsilon(t)} + K\sqrt{T}\,|\, x_\epsilon - x\,|_W\,|\, w_\epsilon\,|_{\mathscr{W}(t)} \\
&\quad \leqslant (M_\epsilon/2)\,|\, w_\epsilon\,|_{\mathscr{V}_\epsilon(t)}^2 + (M/2)\,|\, w_\epsilon\,|_{\mathscr{X}(t)}^2 + (\beta_\epsilon/2)\,|\, w_\epsilon\,|_{\mathscr{W}_\epsilon(t)}^2 \\
&\qquad + (\beta/2)\,|\, w_\epsilon\,|_{\mathscr{W}(t)}^2 + (1/\beta)\,|\, f_\epsilon - f\,|_{\mathscr{W}'(t)}^2 \\
&\qquad + (K^2T/\beta)\,|\, x_\epsilon - x\,|_W^2 + (T\sigma_\epsilon^2/2\beta_\epsilon)\,|\, x_\epsilon\,|_{W_\epsilon}^2
\end{aligned}$$

where, for example,

$$\mathscr{W}_\epsilon(t) = L^2(0, t; W_\epsilon), \qquad \mathscr{W}'(t) = L^2(0, t; W')$$

and the other spaces $\mathscr{W}(t)$, $\mathscr{V}_\epsilon(t)$ and $\mathscr{X}(t)$ are defined analogously. We shall also write

$$\mathscr{W}_\epsilon = \mathscr{W}_\epsilon(T), \qquad \mathscr{W}' = \mathscr{W}'(T)$$

and so forth.

In what follows, C will denote a generic constant independent of ϵ and u. We have from the last inequality and (2.4)

$$\begin{aligned}
&\alpha_\epsilon \mid w_\epsilon(t)|_{V_\epsilon}^2 + \mid w_\epsilon(t)|^2 + \beta_\epsilon \mid w_\epsilon |_{\mathscr{W}_\epsilon(t)}^2 + \mid w |_{\mathscr{W}(t)}^2 \\
&\quad \leqslant C\{\alpha_\epsilon \mid w_\epsilon |_{\mathscr{V}_\epsilon(t)}^2 + \mid w_\epsilon |_{\mathscr{X}(t)}^2 + | f_\epsilon - f |_{\mathscr{W}'}^2 \\
&\quad\quad + \mid x_\epsilon - x |_W^2 + (\sigma_\epsilon^2/\beta_\epsilon) \mid x_\epsilon |_{W_\epsilon}^2\}.
\end{aligned}$$

An application of Gronwall's lemma yields

$$\alpha_\epsilon \mid w_\epsilon(t)|_{V_\epsilon}^2 + \mid w_\epsilon(t)|^2 \leqslant C\{| f_\epsilon - f |_{\mathscr{W}'}^2 + \mid x_\epsilon - x |_W^2 + (\sigma_\epsilon^2/\beta_\epsilon) \mid x_\epsilon |_{W_\epsilon}^2\}.$$

We therefore deduce the estimate

$$\begin{aligned}
&| w_\epsilon |_{L^\infty(0,T;X)} + | w_\epsilon |_{L^2(0,T;W)} \\
&\quad \leqslant C\{| f_\epsilon - f |_{L^2(0,T;W')} + | x_\epsilon - x |_W + (\sigma_\epsilon/\sqrt{\beta_\epsilon}) \mid x_\epsilon |_{W_\epsilon}\}.
\end{aligned}$$

Since

$$u_\epsilon - u = w_\epsilon + (x_\epsilon - x) + (v_\epsilon - v),$$

it remains to prove that

$$\begin{aligned}
&| v_\epsilon - v |_{L^\infty(0,T;X)} + | v_\epsilon - v |_{L^2(0,T;W)} \\
&\quad \leqslant C[\max(\rho_\epsilon/\sqrt{\alpha_\epsilon}, \sigma_\epsilon/\sqrt{\beta_\epsilon})]^{1-\theta} \mid v |_{[Y_\epsilon, Z]_\theta}.
\end{aligned}$$

To prove (3.4), it suffices to prove the following two special cases of (3.4).

(3.5) If $v \in Y_\epsilon = H^1(0, T; V_\epsilon)$ then

$$\begin{aligned}
&| v_\epsilon - v |_{L^\infty(0,T;X)} + | v_\epsilon - v |_{L^2(0,T;W)} \\
&\quad \leqslant C \max(\rho_\epsilon/\sqrt{\alpha_\epsilon}, \sigma_\epsilon/\sqrt{\beta_\epsilon}) \mid v |_{H^1(0,T;V_\epsilon)}.
\end{aligned}$$

(3.6) If $v \in Z = L^2(0, T; W) \cap H^1(0, T; X)$ then

$$| v_\epsilon - v |_{L^\infty(0,T;X)} + | v_\epsilon - v |_{L^2(0,T;W)} \leqslant C \, | v |_Z .$$

In fact, assuming (3.5) and (3.6) to hold let v be given in $L^2(0, T; W) \cap H^1(0, T; X)$ (for example) with $v(0) = 0$ and v_ϵ be the unique solution in $H^1(0, T; V_\epsilon)$ of

$$\mathscr{A}_\epsilon(t)\, v_\epsilon' + \mathscr{B}_\epsilon(t)\, v_\epsilon = v' + \mathscr{B}(t) v, \qquad v_\epsilon(0) = 0.$$

A linear mapping R_ϵ is then defined by setting

$$R_\epsilon v = v_\epsilon - v.$$

Consider the Banach space $L^\infty(0, T; X) \cap L^2(0, T; W)$ with norm defined by

$$| u |_{L^\infty(0,T;X)} + | u |_{L^2(0,T;W)} .$$

(3.5) and (3.6) show that R_ϵ is bounded both as a mapping on $Z = L^2(0, T; W) \cap H^1(0, T; X)$ and as a mapping on $Y_\epsilon = H^1(0, T; V_\epsilon)$ into $L^\infty(0, T; X) \cap L^2(0, T; W)$. It follows from Proposition 2.1 that the restriction of R_ϵ to $[Y_\epsilon , Z]_\theta$ is bounded as a mapping from this space into $L^\infty(0, T; X) \cap L^2(0, T; W)$ with bound

$$M_\theta \leqslant C[\max(\rho_\epsilon/\sqrt{\alpha_\epsilon}\,, \sigma_\epsilon/\sqrt{\beta_\epsilon})]^{1-\theta},$$

where C does not depend on θ. This last estimate is exactly the content of (3.4).

We proceed to the proof of (3.5). From (3.2) and (3.3) we obtain the following equality, valid for almost all t in $[0, T]$:

$$\begin{aligned} &a_\epsilon(t; v_\epsilon' - v', v_\epsilon - v) + b_\epsilon(t; v_\epsilon - v, v_\epsilon - v) \\ &\quad = -r_\epsilon(t; v', v_\epsilon - v) - s_\epsilon(t; v, v_\epsilon - v). \end{aligned}$$

We take the real part and integrate from 0 to t to obtain the estimate

$$\begin{aligned} (\alpha_\epsilon/2) \, | v_\epsilon(t) - v(t) |_{V_\epsilon}^2 &+ (\alpha/2) \, | v_\epsilon(t) - v(t) |^2 + \beta_\epsilon \, | v_\epsilon - v |_{\mathscr{W}_\epsilon(t)}^2 + \beta \, | v_\epsilon - v |_{\mathscr{W}(t)}^2 \\ &\leqslant (M_\epsilon/2) \, | v_\epsilon - v |_{\mathscr{V}_\epsilon(t)}^2 + (M/2) \, | v_\epsilon - v |_{\mathscr{X}(t)}^2 \\ &\quad + (\rho_\epsilon^2/2\alpha_\epsilon) \, | v' |_{\mathscr{V}_\epsilon(t)}^2 + (\alpha_\epsilon/2) \, | v_\epsilon - v |_{\mathscr{V}_\epsilon(t)}^2 \\ &\quad + (\sigma_\epsilon^2/2\beta_\epsilon) \, | v |_{\mathscr{W}_\epsilon(t)}^2 + (\beta_\epsilon/2) \, | v_\epsilon - v |_{\mathscr{W}_\epsilon(t)}^2 . \end{aligned}$$

Therefore,

$$\alpha_\epsilon \mid v_\epsilon(t) - v(t)|^2_{V_\epsilon} + \mid v_\epsilon(t) - v(t)|^2 + \beta_\epsilon \mid v_\epsilon - v \mid^2_{\mathscr{W}_\epsilon(t)} + \mid v_\epsilon - v \mid^2_{\mathscr{W}(t)}$$

$$\leqslant C\{\alpha_\epsilon \mid v_\epsilon - v \mid^2_{\mathscr{V}_\epsilon(t)} + \mid v_\epsilon - v \mid^2_{\mathscr{X}(t)} + (\rho_\epsilon{}^2/\alpha_\epsilon) \mid v' \mid^2_{\mathscr{V}_\epsilon} + (\sigma_\epsilon{}^2/\beta_\epsilon) \mid v \mid^2_{\mathscr{W}_\epsilon}\}.$$

Gronwall's lemma yields

$$\alpha_\epsilon \mid v_\epsilon(t) - v(t)|^2_{V_\epsilon} + \mid v_\epsilon(t) - v(t)|^2 \leqslant C\{(\rho_\epsilon{}^2/\alpha_\epsilon) \mid v' \mid^2_{\mathscr{V}_\epsilon} + (\sigma_\epsilon{}^2/\beta_\epsilon) \mid v \mid^2_{\mathscr{W}_\epsilon}\}.$$

(3.5) follows from the last two inequalities.
To prove (3.6) we start with the equality

$$(v_\epsilon' - v', v_\epsilon - v) + b(t; v_\epsilon - v, v_\epsilon - v)$$
$$= -(v_\epsilon' - v', v) - b(t; v_\epsilon - v, v) - r_\epsilon(t; v_\epsilon', v_\epsilon) - s_\epsilon(t; v_\epsilon, v_\epsilon).$$

Proceeding as before we obtain

$$\tfrac{1}{2} \mid v_\epsilon(t) - v(t)|^2 + \gamma \mid v_\epsilon - v \mid^2_{\mathscr{W}(t)}$$
$$\leqslant \mid v_\epsilon(t) - v(t)| \mid v(t)| + \mid v_\epsilon - v \mid_{\mathscr{X}(t)} \mid v' \mid_{\mathscr{X}(t)} + K \mid v_\epsilon - v \mid_{\mathscr{W}(t)} \mid v|_{\mathscr{W}(t)}$$
$$- \tfrac{1}{2} r_\epsilon(t; v_\epsilon(t), v_\epsilon(t)) + (\rho_\epsilon/2) \mid v_\epsilon \mid^2_{\mathscr{V}_\epsilon(t)} + \sigma_\epsilon \mid v_\epsilon \mid^2_{\mathscr{W}_\epsilon(t)}.$$

Therefore,

$$\mid v_\epsilon(t) - v(t)|^2 + \mid v_\epsilon - v \mid^2_{\mathscr{W}(t)}$$
$$\leqslant C\{\mid v(t)|^2 + \mid v' \mid^2_{\mathscr{X}(t)} + \mid v_\epsilon - v \mid^2_{\mathscr{X}(t)} + \mid v \mid^2_{\mathscr{W}(t)}$$
$$+ \rho_\epsilon(\mid v_\epsilon(t)|^2_{V_\epsilon} + \mid v_\epsilon \mid^2_{\mathscr{V}_\epsilon(t)}) + \sigma_\epsilon \mid v_\epsilon \mid^2_{\mathscr{W}_\epsilon(t)}\}$$

and so, after application of Gronwall's lemma,

$$\mid v_\epsilon(t) - v(t)|^2 + \mid v_\epsilon - v \mid^2_{\mathscr{W}(t)}$$
$$\leqslant C\{\sup_{0 \leqslant s \leqslant t} [\mid v(s)|^2 + \rho_\epsilon \mid v_\epsilon(s)|^2_{V_\epsilon}] + \mid v' \mid^2_{\mathscr{X}(t)} + \mid v \mid^2_{\mathscr{W}(t)}$$
$$+ \rho_\epsilon \mid v_\epsilon \mid^2_{\mathscr{V}_\epsilon(t)} + \sigma_\epsilon \mid v_\epsilon \mid^2_{\mathscr{W}_\epsilon(t)}\}.$$

We also have, for almost all $t \in [0, T]$,

$$a_\epsilon(t; v_\epsilon', v_\epsilon) + b_\epsilon(t; v_\epsilon, v_\epsilon) = (v', v_\epsilon) + b(t; v, v_\epsilon)$$

so that

$$(\alpha_\epsilon/2) \mid v_\epsilon(t)|_{V_\epsilon}^2 + (\alpha/2) \mid v_\epsilon(t)|^2 + \beta_\epsilon \mid v_\epsilon \mid_{\mathscr{W}_\epsilon(t)}^2 + \beta \mid v_\epsilon \mid_{\mathscr{W}(t)}^2$$

$$\leqslant (M_\epsilon/2) \mid v_\epsilon \mid_{\mathscr{V}(t)}^2 + (M/2) \mid v_\epsilon \mid_{\mathscr{X}(t)}^2 + \mid v' \mid_{\mathscr{X}(t)} \mid v_\epsilon \mid_{\mathscr{X}(t)} + K \mid v \mid_{\mathscr{W}(t)} \mid v_\epsilon \mid_{\mathscr{W}(t)}.$$

It follows in the same way as before that

$$\alpha_\epsilon \mid v_\epsilon(t)|_{V_\epsilon}^2 + \mid v_\epsilon(t)|^2 + \beta_\epsilon \mid v_\epsilon \mid_{\mathscr{W}_\epsilon(t)}^2 + \mid v_\epsilon \mid_{\mathscr{W}(t)}^2 \leqslant C(\mid v' \mid_{\mathscr{X}}^2 + \mid v \mid_{\mathscr{W}}^2).$$

Using this last inequality in (3.7), together with (2.4) and the inequality $| v(t)| \leqslant \sqrt{T} \mid v' \mid_{\mathscr{X}}$ we obtain

$$\mid v_\epsilon(t) - v(t)|^2 + \mid v_\epsilon - v \mid_{\mathscr{W}(t)}^2 \leqslant C(\mid v' \mid_{\mathscr{X}}^2 + \mid v \mid_{\mathscr{W}}^2)$$

which proves (3.6).

4. Application

The following illustrates the kind of problem to which the results of the previous section can be applied. The example we have chosen can be extended in several directions, but we do not here attempt maximum generality nor claim to be comprehensive in any sense.

Let Ω be a bounded open set in R^N with boundary Γ. We shall be dealing with the Hilbert spaces (see [10, Chapter 1] for details)

$$H^m(\Omega) = \{u: D^i u \in L^2(\Omega), \mid i \mid \leqslant m\}$$

(m = positive integer) and

$$H^s(\Omega) = [H^m(\Omega), L^2(\Omega)]_\theta, \qquad (1 - \theta)m = s > 0.$$

The norm in $H^s(\Omega)$ denoted by $| \cdot |_{s,\Omega}$; for $s = m$

$$\mid u \mid_{m,\Omega} = \left(\int_\Omega \sum_{|i| \leqslant m} \mid D^i u(x)|^2 \, dx\right)^{1/2}.$$

Let $l \leqslant l' \leqslant m$ ($l < m$) be positive integers and $Q = \Omega \times (0, T)$. We consider (formally) the differential equation

$$[1 + \alpha(\epsilon) A(x, t; D)]\partial u_\epsilon/\partial t + [B(x, t; D) + \beta(\epsilon) \tilde{B}(x, t; D)]u = f_\epsilon(x, t), \qquad (x, t) \in Q, \quad (4.1)_\epsilon$$

with boundary conditions

$$R_k(x; D)\, u_\epsilon = 0, \qquad (x, t) \in \Gamma \times (0, T), \qquad k = 0, 1, \ldots, m-1, \quad (4.2)_\epsilon$$

and initial condition

$$u_\epsilon(x, 0) = u_\epsilon^0(x), \qquad x \in \Omega. \qquad (4.3)_\epsilon$$

$\alpha(\epsilon)$ and $\beta(\epsilon)$ are positive and converge to zero with ϵ and A, B and $\tilde{B}$ are differential operators in x of respective orders m, l and l', with coefficients defined in Q given by

$$A(x, t; D) = \sum_{|i|,|j| \leqslant m} D^i(a_{ij}(x, t)\, D^j), \qquad a_{ij} = \overline{a_{ji}}\,,$$

$$B(x, t; D) = \sum_{|i|,|j| \leqslant l} D^i(b_{ij}(x, t)\, D^j),$$

$$\tilde{B}(x, t; D) = \sum_{|i|,|j| \leqslant l'} D^i(\tilde{b}_{ij}(x, t)\, D^j).$$

We wish to compare the solution of $(4.1)_\epsilon$–$(4.3)_\epsilon$ with the solution of

$$\partial u/\partial t + B(x, t; D)u = f(x, t), \qquad (x, t) \in Q, \qquad (4.1)_0$$

$$R_k(x; D)u = 0, \qquad (x, t) \in \Gamma \times (0, T), \qquad k = 0, 1, \ldots, l-1, \quad (4.2)_0$$

$$u(x, 0) = u^0(x), \qquad x \in \Omega. \qquad (4.3)_0$$

Let $a(t; u, v)$, $b(t; u, v)$ and $\tilde{b}(t; u, v)$ be the sesquilinear forms on $H^m(\Omega)$, $H^l(\Omega)$ and $H^{l'}(\Omega)$, respectively, corresponding to A, B and $\tilde{B}$; for example

$$a(t; u, v) = \sum_{|i|,|j| \leqslant m} \int_\Omega a_{ij} D^j u \overline{D^i v}\, dx, \qquad u, v \in H^m(\Omega).$$

We assume the following:

(4.4) Ω is a bounded open set in R^N of class C^∞.

(4.5) $\{R_k\}_{k=0}^{m-1}$ forms a Dirichlet system of order m on Γ such that R_k has the normal order k. The coefficients in R_k are of class $C^\infty(\Gamma)$.

(4.6) a_{ij}, $\partial a_{ij} \partial t$, b_{ij} and $\tilde{b}_{ij} \in L^\infty(\overline{Q})$.

(4.7) On the closed subspace $H_R^m(\Omega)$ of $H^m(\Omega)$:

$$H_R^m(\Omega) = \{u \in H^m(\Omega)\colon R_k u = 0 \text{ on } \Gamma,\ k = 0, 1, \ldots, m-1\},$$

the form $a(t; u, v)$ is elliptic,

$$a(t; v, v) \geqslant \tilde{\alpha}\, |\, v\, |_{m,\Omega}^2\,, \qquad v \in H_R^m(\Omega), \quad \tilde{\alpha} > 0.$$

(4.8) On the closed subspace $H_R^l(\Omega)$ of $H^l(\Omega)$ the form $b(t; u, v)$ is elliptic,

$$\operatorname{Re} b(t; v, v) \geqslant \gamma \mid v \mid_{l,\Omega}^2, \qquad v \in H_R^l(\Omega), \quad \gamma > 0.$$

(4.9) On the closed subspace $H_R^{l'}(\Omega)$ of $H^{l'}(\Omega)$ the form $\tilde{b}(t; u, v)$ is elliptic,

$$\operatorname{Re} \tilde{b}(t; v, v) \geqslant \tilde{\beta} \mid v \mid_{l',\Omega}^2, \qquad v \in H_R^{l'}(\Omega), \qquad \tilde{\beta} > 0.$$

If we now set

$$a_\epsilon(t; u, v) = (u, v)_{L^2(\Omega)} + \alpha(\epsilon)\, a(t; u, v), \qquad u, v \in H_R^m(\Omega),$$

$$b_\epsilon(t; u, v) = b(t; u, v) + \beta(\epsilon)\, \tilde{b}(t; u, v), \qquad u, v \in H_R^{l'}(\Omega),$$

then one easily verifies that the forms a_ϵ, b_ϵ and b satisfy (2.1)–(2.4) with respect to the spaces

$$V_\epsilon \equiv H_R^m(\Omega), \qquad W_\epsilon \equiv H_R^{l'}(\Omega), \qquad W = H_R^l(\Omega).$$

The quantities α_ϵ, M_ϵ and ρ_ϵ (respectively, β_ϵ, N_ϵ and σ_ϵ) are constant multiples of $\alpha(\epsilon)$ (respectively, $\beta(\epsilon)$). It follows that $(4.1)_\epsilon$–$(4.3)_\epsilon$ $(\epsilon > 0)$ and $(4.1)_0$–$(4.3)_0$ have unique solutions (in the sense of (1.2), (1.3)) u_ϵ and u, respectively, such that

$$u_\epsilon \in H^1(0, T; H_R^m(\Omega)), \qquad u \in L^2(0, T; H_R^l(\Omega) \cap H^1(0, T; W')$$

whenever the data satisfy (3.1), that is,

$$f \text{ and } f_\epsilon \in L^2(0, T; W'), \qquad u_\epsilon^0 \in H_R^m(\Omega), \qquad u^0 \in H_R^l(\Omega). \tag{4.10}$$

THEOREM 4.1. *Assume* (4.4)–(4.10) *hold and that* $u \in H^1(0,T; H^{l+\omega}(\Omega))$ *for some* $\omega > 0$. *Then* $u \in H^1(0, T; [H_R^m(\Omega), H_R^l(\Omega)]_\theta)$ *whenever*

$$1 - \theta < [1/(m - l)] \min(\tfrac{1}{2}, \omega) \tag{4.11}$$

and for all such θ

$$\begin{aligned} &\mid u_\epsilon - u \mid_{L^\infty(0,T;L^2(\Omega))} + \mid u_\epsilon - u \mid_{L^2(0,T;H^l(\Omega))} \\ &\quad \leqslant C\{[\max(\sqrt{\alpha(\epsilon)}, \sqrt{\beta(\epsilon)})]^{1-\theta} [\mid u \mid_{H^1(0,T;H^{(1-\theta)m+\theta l}(\Omega))} \\ &\qquad + \mid u^0 \mid_{(1-\theta)m+\theta l,\Omega}] + \mid f_\epsilon - f \mid_{L^2(0,T;W')} \\ &\qquad + \mid u_\epsilon^0 - u^0 \mid_{l,\Omega} + \sqrt{\beta(\epsilon)} \mid u_\epsilon^0 \mid_{l',\Omega}\}, \end{aligned}$$

where C is independent of ϵ, u and θ.

Remarks. Since

$$[H_R^m(\Omega), H_R^l(\Omega)]_\theta \subset [H^m(\Omega), H^l(\Omega)]_\theta = H^{(1-\theta)m+\theta l}(\Omega),$$

each term in the inequality has a meaning. Also, given $\omega > 0$, the condition $u \in H^1(0, T; H^{l+\omega}(\Omega))$ will be satisfied if u_0, f and the coefficients in B are sufficiently smooth (depending on ω) and f and u_0 satisfy certain (necessary) compatibility relations (see [10, Theorem 4.6.2 and Proposition 4.2.3]).

For $s > 0$, set

$$H_R^s(\Omega) = \{u \in H^s(\Omega); R_k u = 0 \text{ on } \Gamma \text{ for } k < s - \tfrac{1}{2}\}.$$

Under the norm $|\cdot|_{s,\Omega}$, $H_R^s(\Omega)$ is a closed subspace of $H^s(\Omega)$. To prove Theorem 4.1, we use the following result of Grisvard [5] (c.f. [10, Chapter 4, Section 14.5]).

PROPOSITION 4.1. *Assume* (4.4) *and* (4.5) *and that* $(1-\theta)m + \theta l \neq$ *integer* $+ \frac{1}{2}$. *Then*

$$[H_R^m(\Omega), H_R^l(\Omega)]_\theta = H_R^{(1-\theta)m+\theta l}(\Omega)$$

with equivalent norms.

Proof of Theorem 4.1. Condition (4.11) implies $l < (1-\theta)m + \theta l < l + \frac{1}{2}$ and therefore $(1-\theta)m + \theta l \neq$ integer $+ \frac{1}{2}$. We also note that the regularity hypothesis on u, together with the fact that $u(t) \in H_R^l(\Omega)$ $(0 \leqslant t < T)$, implies $u \in H^1(0, T; H_R^l(\Omega))$ since $R_k \in \mathscr{L}(H^s(\Omega), H^{s-k-1/2}(\Gamma))$, $s > k + \frac{1}{2}$. It follows from Proposition 4.1 that $u \in H^1(0,T; [H_R^m(\Omega), H_R^l(\Omega)]_\theta)$ provided

$$(1-\theta)m + \theta l - \tfrac{1}{2} < l \qquad \text{and} \qquad (1-\theta)m + \theta l \leqslant l + \omega,$$

that is, provided θ satisfies (4.11). The estimate of the theorem now follows from Corollary 3.1.

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